

How to understand the tunneling in attosecond experiment?

Bohr-Einstein photon box Gedanken experiment, tunneling time and the wave particle duality

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The measurement of the tunneling time (T-time) in today's attosecond and strong field (low-frequency) experiments, despite its controversial discussion, offers a fruitful opportunity to understand time measurement and the time in quantum mechanics. In addition, as we will see in this work, a related controversial issue is the particulate nature of the radiation. Different models used to calculate the T-time will be discussed in this work in relation to my model of real T-time, Phys. Rev. **92**, 052118 (2015), where an intriguing similarity to the Bohr-Einstein photon box *Gedanken experiment* was found. The tunneling process itself is still not well understood, but I am arguing that a scattering mechanism (by the laser wave packet) offers a possibility to understand the tunneling process in the tunneling region. This is related to the question about the corpuscular nature of light which is widely discussed in modern quantum optics experiments.

Keywords: Attosecond physics, tunneling time and time measurement in attosecond experiments, time-energy uncertainty relation, time and time-operator in quantum mechanics, Bohr-Einstein's *photon box Gedanken experiment*, multiphoton processing, Compton scattering, wave-particle duality.

I. INTRODUCTION

There is no doubt that the advent of 'attophysics' opens new perspectives in the study of time resolved phenomena in atomic and molecular physics [1–5], the tunneling process and the tunneling time (T-time) in atoms and molecules [6–10]. Attosecond science concerns primarily electronic motion and energy transport on atomic and molecular scales and is of fundamental interest to physics in general. The time-energy uncertainty relation (TEUR) receives a new breath due to the actual problems of quantum information theory and impressive progress of the experimental technique in quantum optics and atomic physics [2, 11]. In my previous work [10], I showed that using the TEUR (precisely that time and energy are conjugate variables) leads to a nice relation to determine the T-time in good agreement with the experimental finding in the attosecond experiment (for He atom) [9], (1 attosecond = 10^{-18} second). The T-time and time itself in quantum mechanics (QM) are controversial, and there is still common opinion that time plays a role essentially different from the role of position in quantum mechanics (although it is not in line with special relativity, [12]) and that time is a parameter, like a classical

Newtonian time quantity, and hence does not obey an ordinary TEUR. Nevertheless, Hilgevoord concluded in his work [13] that when looking to a time operator a distinction must be made between the universal time coordinate t , a c-number like a space coordinate, and the dynamical time variable of a physical system situated in space-time; i.e. clocks. Accordingly in [10, 14] it was shown that the T-time is intrinsic, i.e. dynamically connected to the system (internal clock) after the classification of Busch [15, 16] and [17] (chap. 3). Fortunately, Bauer's introduction of a self-adjoint dynamical time operator in Diracs relativistic quantum theory [12, 18], supports the results of [10]. In [18, 19] Bauer concluded that the dynamical time operator provides a straightforward explanation (within standard relativistic quantum mechanics) of the T-times, which is measured in the photoionization experiments, compare the discussion in [14]. In this respect, Bauer also rejects the claim of Dodonov [11] that no unambiguous and generally accepted results have been obtained for the time operator [18, 20]. Moreover, Bauer showed [18] that the Mandelstam-Tamm uncertainty associated with the observable \hat{T} largely overestimates the internal time standard uncertainty as already discussed by Kullie [14].

A similar controversial issue to the time issue (and the T-time and TEUR) is the wave-matter duality and the partic-

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ulate nature of the light [21, 22], since the Einstein hypothesis of the quanta as a carrier of $h\nu$ based on the Planck hypothesis of the quantization of the energy $E = h\nu$. The term photon was given by G. N. Lewis in 1926 [23], and indeed the corpuscular hypothesis originally stems from Newton. As we will see in this work, the two issues closely appear in today's attosecond experiments (ASEs). Indeed, since the appearance of QM time was controversial, the famous example is the Bohr-Einstein weighing *photon box Gedanken experiment* (*BE-photon-box-GE*). In [10] I showed with a simple tunneling model that the tunneling in the attosecond experiment is intriguingly similar to the *BE-photon-box-GE*, where the former can be seen as a realization to the later, with the electron as a particle (instead of the photon) and an uncertainty in the energy being determined from the (Coulomb) atomic potential due to the electron being disturbed by the field F , instead of (the photon) being disturbed by the weighting process and, as a result, an uncertainty in the gravitational potential [24], as shown by the famous proof of Bohr (see for example [25] p. 132) to the uncertainty (or indeterminacy) of time in the *BE-photon-box-GE* [15, 16, 24].

The T-time and the tunneling process itself in the ASEs are hot debated, and the later is still rather unresolved puzzle. In the (low-frequency) ASEs the idea is to control the electronic motion by laser fields that are comparable in strength to the electric field in the atom. In today's experiments usual intensities are $\sim 10^{14} \text{ Wcm}^{-2}$, for more details we refer to the tutorial [4–6, 26, 27]. In the majority of phenomena in attosecond physics, one can separate the dynamics into a domain “inside” the atom, where atomic forces dominate, and “outside”, where the laser force dominates, a two-step semiclassical model, pioneered by Corkum [28]. Ionization as the transition from “inside” to “outside” of the atom plays a key role for attosecond phenomena. A key quantity is the Keldysh parameter [29],

$$\gamma_K = \frac{\sqrt{2I_p}}{F} \omega_0 = \tau_K \omega_0, \quad (1)$$

where I_p denotes the ionization potential of the system (atom or molecule), ω_0 is the central circular frequency of the laser pulse or the laser wave packet (LWP) and F , throughout this work, stands for the peak electric field strength at maximum, and τ_K denotes the Keldysh time. Hereafter in this work (unless it is clear), atomic units are used, where $\hbar = m = e = 1$, the Planck constant, the electron mass and the unit charge are all set to 1. At values $\gamma_K > 1$ one expects predominantly photo-ionization or multiphoton ionization (MPI), while at

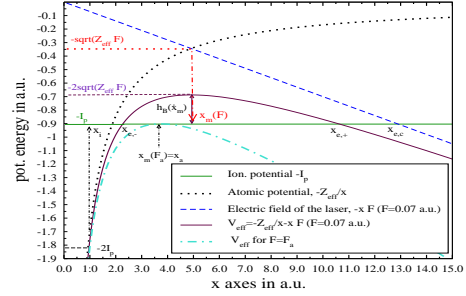


FIG. 1. (Color online) Graphic display of the potential and the effective potential curves, the two inner and outer points $x_{e,\pm} = (I_p \pm \delta_z)/2F$ and the barrier width $d_B = x_{e,+} - x_{e,-}$, the “classical exit” point $x_{e,c} = I_p/F$ and $x_m(F) = (Z_{eff}/F)^{1/2}$ the position at the maximum of the barrier height $h_B(x_m)$, (note $x_a = x_m(F = F_a)$), see text.

$\gamma_K < 1$ (field-)ionization happens by a tunneling process (for $F < F_a$), which means that the electron does not have enough energy to ionize directly, and therefore it tunnels (or tunnel-ionizes) through a barrier made by the Coulomb potential and the electric field of the laser pulse and escapes at the exit point to the continuum, as shown in fig 1, see the following section. We pay attention to one important case study in attosecond physics, the T-time measurement in ASE performed by Keller [7–9] and we will refer to it as the Keller ASE (KASE). In this experiment an elliptically polarized laser pulse is used with $\omega_0 = 0.0619 \text{ au}$ ($\lambda = 736 \text{ nm}$), the ellipticity parameter $\epsilon = 0.87$, while the electric field strengths are in the range $F = 0.04 - 0.11$ and for He atom $I_p = 0.90357 \text{ au}$.

II. THE TUNNELING TIME

Usually the tunneling process in the ASEs is explained by a simple picture, like the one shown in fig 1 for the He atom. It is based on the strong field approximation (SFA) or Keldysh-Faisal-Reiss approximation [29–31]; for an introductory review see [32]. This simple picture is very useful in explaining the experiment, although it is strictly true only in length gauge [33–35]. Physical quantities are independent of a gauge transformation as long as exact equations (or the same orders of approximation [36]) are employed. Indeed, the length gauge or the dipole approximation of the interaction Hamiltonian due to the Göppert-Mayer gauge-transformation (when long-wavelength approximation applied) also has the advantage that it leads to an expression for the interaction energy involving mathematical quantities, each of which has a ready physical interpretation [37].

In the tunneling process in the ASEs, according to the SFA

(see also [38]), the electron tunnels and escapes the barrier region at the exit point $x_{e,+}$, see fig 1, with approximately zero kinetic energy, more precisely the electron velocity along the field direction is zero and negligible in the other directions. In [10] (hereafter Kullie model) we showed that the uncertainty in the energy, which is related to the height of the barrier $h_B(x_m)$, can be quantitatively discerned from the atomic potential energy at the exit point $\Delta E \sim |V(x_e)| = |-Z_{eff}/x_e|$ for arbitrary strengths $F \leq F_a$, where Z_{eff} is the nuclear effective charge and $F_a = I_p^2/(4Z_{eff})$ is the atomic field strength [39, 40] [41]. With the TEUR, $\Delta E \cdot \Delta T \leq 1/2$, one obtains the symmetrical (or total) T-time [10]:

$$\tau_{T,sym} = \tau_{T,i} + \tau_{T,d} = \frac{1}{2} \left(\frac{1}{(I_p + \delta_z)} + \frac{1}{(I_p - \delta_z)} \right), \quad (2)$$

where $\delta_z = \sqrt{I_p^2 - 4Z_{eff}F}$. The relation in eq 2, besides the mathematical simplicity, aids a conceptual reasoning [10, 14] and the discussion further below. The physical reasoning of this relation is the following: the barrier itself causes a delaying time $\tau_{T,d}$, which is the time delay with respect to the ionization at atomic field strength F_a , where the barrier is absent (i.e. the barrier height, the barrier width and δ_z are zero), it is the time duration to pass the barrier region (between $x_{e,-}$, $x_{e,+}$) and escape at the exit point $x_{e,+}$ to the continuum, for more details see [10]. The first term $\tau_{T,i}$ in eq 2 is the time needed to reach the entrance point $x_{e,-}$ from the initial point x_i , compare fig 1. At the limit $F \rightarrow F_a$, $\delta_z \rightarrow 0$ and the total time becomes the ionization time $\tau_{T,sym} = \frac{1}{I_p}$ at the atomic field strength F_a .

The T-time is a controversial issue from different points of view, as we will discuss in the following subsections.

A. A real or an imaginary quantity

First, the most reasonable argument is that that T-time is a real quantity, as clarified by Steinberg [17] (chap. 11) and Büttiker [17] (chap. 9), although many authors claim, and it is widely accepted, that it is an imaginary quantity [42, 43], [44] (and [17] chap. 7). The imaginary tunneling time point of view relies on the fact that the tunneling is classically forbidden. Although that is important, it is not instructive, in the case it obscures an insight and otherwise accessible conceptual understanding [45]. One has to mention that a complex time point of view (i.e. real and imaginary parts) would not change the conclusion of our works, because the crucial point in our discussion is a real part of the T-time. In agreement

with the real T-time point of view, with the real time of the FPI description of [9] (although Sokolovski [44], [17] chap. 7, argued that T-time described by the FPI is imaginary), and the entropic formulation (or the statistical approach) of (real) T-time of Demir [46], the relation in eq 2 presents a real T-time model which explains the T-time in KASE in a good agreement with the experimental finding [10, 14], compare fig 2, 3. Although the model is simple, it is important in the quest to answer the question: how to understand the T-time and tunneling process, the time measurement and the time in quantum mechanics [14].

Secondly, the treatment in [10, 14] benefits from the internal (intrinsic or dynamical) time point of view (internal clock, [17] chap. 3), this requires one to choose a reference point [2, 13, 24], which can be at best determined by the (natural) internal properties of quantum mechanical systems (e.g. ionization potential I_p). This is important as it enables one to identify or map the internal delay time (a time interval) as a delay time in an external clock without, certainly, contraction or dilation of the time (scale unity) interval of the clocks (i.e. no effects like in the relativistic theory.) We also note that the internal clock or the intrinsic time point of view is similar to what occurs in special relativity, where a moving particle has its own time in its inertial frame, which differs from the time from the viewpoint of other inertial frames, as discussed in [14]. Finally we mention that some authors [42, 43] use the notation $t_s = t_R + i t_T$, where t_s refers to the solution of the saddle point equation, t_R or the real part of t_s denoted as the ionization time (after tunneling) and the imaginary part t_T as the T-time. The partition of t_s this way, in a real part for ionization and an imaginary part for the tunneling, lets some questions be opened. We discuss this in the next subsec II B. It is worthwhile to mention (see discussion in [10]) that in eq 2, δ_z becomes imaginary $\delta_{im,z} = i \delta_z$ (i the imaginary number) for $F \geq F_a$ and that is the above-threshold region.

B. Many different approaches

The scattering theory concept is widely used to calculate the T-time from the scattering time (for example Büttiker-Landauer or Pollak-Miller time, for details see [49, 50]. However, following Collins [49] this is not justified. Recently Landsman [9] showed that these time approximations are in disagreement with the experimental finding of KASE, and that the FPI with a coarse graining procedure fits well with the

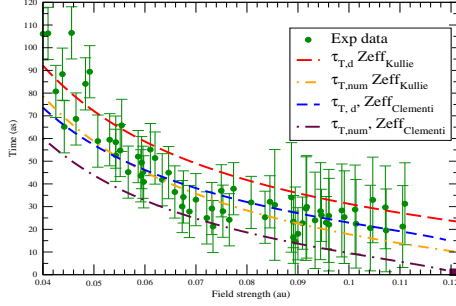


FIG. 2. (Color online) Graphic display of tunneling time vertical axes (in attosecond) vs field strength F in au . $\tau_{T,d}$ eq 2 (red $Zeff_{Kullie}$, blue $Zeff_{Clementi}$, and τ_{num} eq 3 (orange $Zeff_{Kullie}$, brown $Zeff_{Clementi}$). $\tau_{T,d}, \tau_{num}$ are plotted with F as a free variable. Experimental values from [9].

KASE measurement data. Collins's most critical point [49], which addresses the question: how long does it take for an electron to tunnel through a barrier, was the conclusion that the scattering time, defined through the inverse of the transition probability matrix, is not related to the dynamical transport behavior of the electron's tunneling. It is associated with the finite lifetime and decay of metastable states, in this case being the tunneling electrons treated as quasi-particles which are decaying from a state on one side of the potential barrier into another state on the opposite side of the barrier. Since this quantifies the decay of metastable states and not the transit time of an electron across a barrier, this time can be quite large and is a steady-state picture which does not reflect the dynamical nature of the tunneling particle(s). And according to Collins, it can be seen on the basis of the time-independent picture and by analogy with a particle decay, that the scattering time represents a mean time in which a certain likelihood of a tunneling event may take place. This does not reflect the actual time of a tunneling process [49]. A similar point of view was considered by Fock and Krylov [51], in regards to the lifetime and TEUR, see discussion by Aharonov [52].

Collins then showed (using a Gaussian wave packet in the simulation), that the phase-time is overall the best one to use when momentum skewing of the initial wave packet is not significant. Unfortunately, in [14] we showed first that the phase time in attosecond experiment leads to the Keldysh time τ_K , eq 1 (an approximation that neglects the effect of the core potential, and could be important for the evolution of the wave packet), which is far from the experimental finding [9]. And second that a time-delay requires us to choose a reference (point); delays in numerical simulation can refer in principle to any arbitrarily chosen reference [2, 14]. Indeed, because $\tau_{T,d}$ in

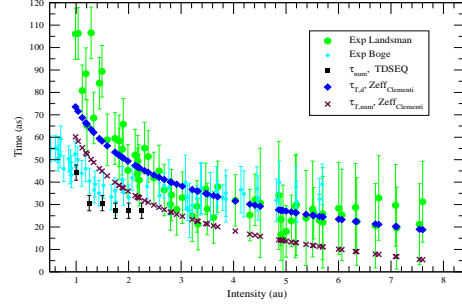


FIG. 3. (Color online) Graphic display of tunneling time vertical axes (in attosecond) vs the intensity in au . $\tau_{T,d}, \tau_{num}$ are plotted at the experimental value of [9]. $\tau_{T,d}$ (upper, blue) eq 2, and τ_{num} (lower, brown) eq 3 with $Zeff_{Clementi}$, see text. The black squares are numerical integration of TDSEQ form [47], data kindly sent by I. A. Ivanov. The points with error bars are experimental data, green (large circles) from Landsman [9] and the light blue (small circles) from R. Boge [48], data kindly sent by A. S. Landsman and R. Boge.

eq 2 presents a delay time relative to the limit at atomic field strength $\tau_{T,d}(F = F_a) = 1/(2I_p) \equiv \tau_{T,d,F_a}$, by subtracting the latter from former we get

$$\begin{aligned} \tau_{T,d} - \tau_{T,d,F_a} &= \frac{1}{2(I_p - \delta_z)} - \frac{1}{2I_p} = \frac{(\frac{\delta_z}{I_p})}{2(I_p - \delta_z)} \equiv \tau_{num} \quad (3) \\ &= \frac{\delta}{2Ip^2} + \dots \frac{\delta^k}{2Ip^{k+1}} \dots = \frac{1}{2Ip} \sum_{k=1}^{\infty} \left(\frac{\delta}{Ip} \right)^k \end{aligned}$$

where in the second line an expansion of the form $(1 - x)^{-1} = 1 + x + x^2 + \dots$, $x = (\delta/Ip)$, is used, and $1/(2Ip)$ is the zero order term ($k = 0$). In fig 2 the T-time $\tau_{T,d}$ of eq 2 and τ_{num} of eq 3 are compared with experimental data of [9], the field strength F is the free variable and runs between 0.04 – 0.12, the small dark square (at the right lower corner) marks F_a , where τ_{num} becomes zero and $\tau_{T,d} = 1/2Ip$ as already mentioned. The crucial difference between τ_{num} and $\tau_{T,d}$ is that the former tends to zero at atomic field strength ($\delta_z(F_a) = 0$), $\lim_{F \rightarrow F_a} \tau_{num} = 0$, which can happen only numerically, whereas quantum mechanically $\lim_{F \rightarrow F_a} \tau_{T,d} = \frac{1}{2Ip}$ is the second part of the total ionization time, eq 2, at atomic field strength ($\tau_{T,sysm} = \frac{1}{2Ip} + \frac{1}{2Ip} = \frac{1}{Ip}$), as discussed in [10]. This can be seen in fig 2 for $Zeff = 1.6875$ of Clementi [53], where $F_a \approx 0.12 au$, which has to be chosen in this region and matches the experiment. $Zeff = 1.375$ of Kullie ($F_a \approx 0.14 au$) [54] is the better choice in the region for small field strengths, for detailed discussion see [10].

In his work [19], titled the problem of time in quantum mechanics, Bauer mentioned that time interred in the time dependent Schrödinger equation (TDSEQ) has to be identified with a parameter (parametric time t_{parm}) that corresponds to the time coordinate of the laboratory frame of reference as claimed by Briggs [55]. Although τ_{num} differs qualitatively

from a τ_{parm} (defined below), because it is constructed from the dynamical (internal) time of the systems, it looks to be identical with the parametric time τ_{parm} , i.e. one can write

$$\tau_{parm} = \tau_{num} = t_{T,d} - \frac{1}{2I_p}, \quad (4)$$

where τ_{parm} denotes a time interval such that $\tau_{T,d} = \tau_{parm} + 1/(2I_p)$ and $1/(2I_p)$ is the ionization time (at atomic field strength F_a). Thus, the identification of the time in the TDSEQ as a parametric time indicates the lack of a proper mapping of the time from the internal (time-frame) to an external time (time-frame) and not an inherent property of the time in TDSEQ i.e. in quantum mechanics, in agreement with Hilgevoord point of view and Bush classification as mentioned in sec.I. Thus; so τ_{num} is identical with the time interring the TDSEQ, when mapping the dynamical (internal) time to an external (laboratory) time by ignoring the reference point (i.e. neglecting $1/(2I_p)$), as clearly seen in eq 4 (and in figs 2, 3, see below.) Whereas $\tau_{T,d}$ counts the internal time (interval) and transforms it to the external time (interval, measurement data) when the electron moves from inner to the outer region (the tunneling or ionization process) due to the interaction with the laser field. Contrary to the (parametric) time of measurement thought by Aharonov [52], that the time of measurement belongs to the observing apparatus, which has been corrected in his recent work [24]. Note a time interval refers to a clock, which measures the time quantity with respect to a reference point, whereas a time variable denotes a time quantity of any type, see subsec II A 2nd part. And as discussed in [14] perhaps only the classical Newtonian time is a parametric (external non-dynamical) type of time. In fig 3 the experimental data of [48] (light blue) and [9] (green) are displayed, they are from the same set of experimental data, but a renewed calibration procedure of the field strength was used in the later. One sees that the T-time points resulting from numerical integration of the TDSEQ of Ivanov [47] (black squares), lie somehow below the experimental data of [48], after shifting the experimental data (done by Landsman) one sees that our τ_{num} eq 3, where the inserted F values are taken from the experiment, lies below the experimental data of Landsman [9] in a similar way the points of Ivanov [47] lie below the experimental data of Boge [48] [56].

Nevertheless, Collins conclusion shows that the usual scattering concepts when used for the ASEs are best suited to the MPI region, where momentum skewing of the electronic wave packet is small [49], compare subsec II C. Therefore, it is not surprising that Torlina et al, when using ARM (analytical R-

matrix theory) to calculate the T-time in the ASEs (the attoclock), argued that the ARM requires sufficiently thick tunneling barriers [42, 57, 58]. Torlina et al calculated the T-time of the (field-)ionized electron from the ground state of the Hydrogen atom, which is exposed to an attosecond laser pulse [42]. They also claim that T-time is imaginary (see II A) and no real tunneling delay time is associated with the tunnel-ionization [42], i.e. for $F < F_a$, and optical tunneling is instantaneous. Instantaneous here means that a real T-time is zero, although it should, in fact, represent the time of a real dynamical process and that at the tunnel exit the electron is far from the nucleus, i.e. the tunnel distance is not negligible as confirmed experimentally by [59]. However, it can be easily shown that the request of sufficiently thick tunneling barriers corresponds to the region of small field strength, for which (at optical frequencies) most likely $\gamma_K \sim \frac{1}{F} > 1$, see fig 1 and eq 1, and that is the MPI region, where tunneling is not probable. In addition, it could be that the imaginary T-time of Torlina et al is related to the phase time of Collins [49] as discussed above, see also [14]. In this case one expects that the phase time of the electronic wave packet, like the phase velocity, has no real physical significance, although it can be viewed as a characteristic of the evolution of the wave packet in the sense described by Messiah [60], see discussion in [14].

1. In depth discussion

A point, which I think is important here, is that the calculations of Torlina et al should be compared with the time-of-arrival ([17] chap 3, [61, 62]), [63]. The later concerns the subsequent propagation of the tunnel-ionized electron (see below). One notes that in attosecond and strong-field science the (tunnel-)ionization process cannot be understood in its conventional form known in quantum optics [64]. Important here (compare [42]): First the calculation of the ionization time of Torlina et al (hereafter Torlina model) relies on the dynamic interaction of the outgoing electron with the remaining core, i.e. the effects of the long-range potential (starting at the exit point), and that the ionization is not yet completed at the moment the electron exits the tunneling barrier (note the different definitions of the barrier, see below). This looks similar to the concept of the time-of-arrival method and the observable-type of time after the classification of Busch ([17] chap 3, where in this case the atom acts as a source (confining the electronic wave packet up to $Re(t) = 0$) [17] chap. 1, and where the real time counting, as Torlina et al also argued, starts at the tunnel exit of their model (which, however, defines the border between inner and outer regions.) Indeed, the potential at the exit point $x_{exit} = x_{e,+}$ defines the energy uncertainty in the model of Kullie [10], here we indicate a common point (consideration) between the two models, and that the potential energy at the exit point is the central quantity, although the exit points $x_{e,+}$ and $x_{e,c}$, and the assignment of the inner and outer regions in the tunneling process are different in the two models, see fig 1. We argue that, the electron escapes the tunnel exit $x_{e,+}$ (where the energy uncertainty is defined), $x_{e,+} > x_{e,-} > x_i$ (x_i is the initial position, see fig 1) is real and hence it is reached by a real time, and in the outer region (after

$x_{e,+}$, where according to the SFA the electron is free) the effect of the long range potential decreases rapidly, i.e. the measurement data of [9] corresponds to the crossing of the barrier region (between $x_{e,-}$ and $x_{e,+}$) when the tunneling process occurs [9, 10], in the length gauge picture. Our real T-time $\tau_{T,d}$ represents a delay time with respect to the natural reference point (i.e. ionization at the atomic field strength F_a), which is certainly real, since the exit point $x_{e,+} \geq x_{e,-} > x_i$ (equality for F_a) is real, and quantum mechanically such a tunneling/ionization dynamical time can not be zero or purely imaginary.

And second, the authors of [42] claim that although the measured quantity (the electron momentum) is real, the trajectories in the ARM method are not classical, in the sense that the trajectories have both real and imaginary components all the way to the detector, where they claim that the real part of the trajectory starts near the origin without an explicit definition [65]. This, in turn, shows that in the tunneling process, real and imaginary components of a trajectory and (hence) real and imaginary components of time can exist (quantum mechanically) in both the inside and outside regions (under the barrier and after tunneling, despite the differences in the definition of these regions). Thus the barrier region is not necessary captured solely by an imaginary time component although it is classically forbidden.

This leads to the conclusion, that the partition $t_s = t_R + it_T$ mentioned above in subsec IIA, i.e. an imaginary part t_T for tunneling and a real part t_R for ionization after the tunneling, is at least not unique if acceptable, apart from the fact that time delay requires one to choose a reference system [14, 27]. This partition is only in line with the classical point of view that the dynamics under the barrier is classically forbidden, hence no real T-time can exist for the dynamics in this region, i.e. the tunneling process, although quantum mechanically tunneling is a possible physical process. The partition goes back to Perelomov et al [38, 66, 67] (hereafter PPT model), it was based on the argument that the (semi-)classical trajectory is determined by the initial conditions $x(t_0) = 0, v(t_0) = ik, k = (2I_p)^{1/2}$, where t_0 is the time when the field reaches its maximum. The time was chosen such that the origin for the real time is the instant when $v_{exit} = v(t_R = 0) = 0$, i.e. $t_R = 0$ when the particle emerges from the barrier, but at the same time [67] the exit point is real, $x_{exit} = x(t_R = 0) = x_c \neq 0$, where $x_c = I_p/F \gg x_i$ is usually called the classical exit point. This makes the assumption that the T-time is a purely imaginary quantity questionable. Tunneling is a quantum mechanical effect and to my best knowledge there exists no physical restriction, which forces us to assume that the dynamical time to overcome the barrier region, should be a purely imaginary quantity, which is a classical standpoint and did not reflect the quantum nature of the electron's motion in the barrier region, where certainly the energy conservation should not be violated, it turns out that a full understanding of the tunneling process in the ASEs is still waiting.

We note that in the PPT model, used by Torlina et al [42], one defines the inner region through the assumption that the approximate wave function ψ_{in} can be substituted by the (field-free) ground-state wave function [68]. From this one sees that the border between the inner and outer regions in this model is comparable with the entrance point $x_{e,-} = (I_p - \delta_z)/(2F) \approx I_p/(2F) = \tilde{x}_e$ of the model of Kullie [10, 14] as seen in fig 1. For $F = F_a, x_a = x_{e,\pm} = I_p/2F_a$, this case (ionization as $\delta_z = 0$) matches to the real part time as defined by Torlina, however x_a differs from the classical exit point $x_{e,c} = I_p/F = 2x_a$ assumed in Torlina model. As a side note, this means that $\tau_{T,i} = 1/(2I_p + \delta)$ can be chosen as an initial time $t_0 = \tau_{T,i}$ for the Torlina model, see discussion below. \tilde{x}_e is also compatible with their choice of the radius of the sphere separating the inner and outer regions $1/\kappa = (2I_p) < a < I_p/F$ [57]. That means that the so-called under-the-barrier region as given in fig 1 (between $x_{e,-}$ and $x_{e,+}$ of the Kullie model) belongs roughly to the continuum (or outside region) of the Torlina model, although the model

assumes that $x_{e,c} = I_p/F$ is roughly corresponds to the exit of the tunneling barrier. Hence, the tunneling process has a different meaning between the two models, where Torlina et al define it by an imaginary T-time, and by real and imaginary trajectories in the inside and outside regions, where “tunnel-exit” is a complex integrable singularity point of the potential. This can be compatible with the velocity gauge, where the barrier is absent, or not a physical one (the barrier region is crossed in an imaginary time elapse). Indeed, as mentioned above the definition of the barrier and the barrier regions in attosecond physics are gauge dependent [33]. This suggests (but it needs a scrutiny to be accepted) that the Torlina model (when applied to the tunneling region $\gamma_K < 1$) is quantitatively equivalent to the model of Kullie (provided both models have the same orders of approximation, see discussion below about the initial time), although the two models are completely different concepts, and the length gauge (used in Kullie model) has the advantage of presenting a clear physical picture [14, 18], because no physical quantity corresponds to the vector potential in the velocity gauge. Note, Torlina et al also concluded that the total ionization rate depends only on what happens to the electron while it is tunneling [65], hence the confusion is mainly due to the different definitions of the tunneling process and the barrier region or the inside and outside regions.

Another point is that the work of Torlina et al is an operative concept (compare below subsec IIC) it does not touch the concept of time (and its controversial discussion) in quantum mechanics. Indeed, it assumes implicitly a parametric time, point of view, compare eq 4. It is worthwhile to mention that there exists classical procedures, which are quantitatively equivalent to the Torlina model, e.g. the propagating of classical trajectories within a two-step model or (CCSFA) which involves classical Monte Carlo-type simulations [65]. In the Kullie model there is no complicated calculation with a wave function, which is, no doubt, an advantage of the Torlina model. Also no trajectories (real or imaginary) are used, the calculations are achieved in time-energy space based on the TEUR. However, the geometry of the barrier is needed. It covers the time and T-time concepts in quantum mechanics, and makes use of, or offers a qualitative or quantitative connection to, the different fundamental issues of the quantum mechanics, i.e. the BE-photon-box-GE, the double slit experiment and the TEUR.

Finally, the situation looks similar to the equivalence between my model and the FPI description of [9] as mentioned in subsec IIA. At this point, one has to mention that a model called an entropic formulation of T-time [46] exists, the model is also based on the TEUR and the authors claim that their model describes a real T-time, which is in good agreement with Kullie model and the FPI model of [9]. However, crucial points still have to be clarified, e.g. choosing the reference system to calculate the delay time, where possibly it is one of the sources of the negative time in the Torlina model [42] for $F > F_a$ (F_a is the atomic field strength). Indeed, one could consider it as an artifact effect, because Torlina et al evaluate their time starting at the peak of the spectrum, regardless the natural parameters of the system under consideration (the Hydrogen atom), i.e. the ionization potential (possibly with a Stark shift) and the atomic field strength, where the Above-threshold-ionization process starts. The initial real time zero assumption $t_0 = 0$ at the peak intensity, in the attoclock of the Torlina model was criticized by Zimmerman et al, supplemental information [69]. In my understanding, adding the term $\tau_{F_a} = 1/(2I_p)$, see eqs 3, 4, to the Torlina model, can remedy the initial time problem (and the negative time), which enables one to take the system parameters in account and counts the delay (the real time part t_R of t_s) relative to the ionization time τ_{F_a} at atomic field strength. Using the initial time $t_0 = 0$ at the peak of the spectrum is equivalent to eq 4 (compare fig 2, 3), thus the real part of the calculated time by Torlina $t_R^{Torlina} = \tau_{num}$ or in other words, one obtains for the time of the Torlina model

$$Re(\tau_s) \equiv t_R^{Torlina} + t_0 = \tau_{num} + t_0 = \quad (t_0 = \tau_{F_a})$$

with $t_0 = \tau_{d,F_a} = 1/(2I_p)$. $Re(\tau_s)$ is then corresponds to the actual T-time $\tau_{T,d}$, see eq 2. This becomes more apparent, when one compares the result of the numerical integration of the TDSEQ (black squares) with $\tau_{T,d}$, τ_{num} in fig 3.

Another reasonable possibility, to adjust the choosing of the initial time, is to consider $x_{e,-}$ as the initial position where the process (of tunneling or ionization) starts (which differs from the interaction instant, the first step see discussion in [10]) and thus to consider the time to reach $x_{e,-}$ as the initial real time at the peak intensity, and hence $\tau_{T,i} = 1/(2(I_p + \delta_z)) = t_0$, eq 2, serves as a real initial time for the counting of the real time by the Torilna model. The importance of this follows from the fact that only after the interaction [10], the propagation vector of the electron wave function is identical to the propagation vector of the optical vector potential. In addition, it is compatible, as discussed above, with the division inside (ψ_{in}) and outside of the barrier region at the point $x_{e,-}$ in connection to the Torlina approximation. In other words $\tau_{T,i} = t_0$ can be considered as the real part of the T-time that correspond the imaginary part of the T-time of Torlina et al in the inside region, or the above mentioned expression of t_s should be written in the form

$$\tau_s = i\tau_R^{Torlina} + t_0 + i t_T, \text{ with } t_0 = \tau_{T,i} = 1/(2(I_p + \delta_z))$$

i.e. at the start (after the first step [10], the interaction step) the pointer of the attoclock points at $t_0 = \tau_{T,i}$. At $F = F_a$ this become $t_0 = \tau_{F_a} = 1/(2I_p)$, $\delta_z = 0$, where the difference to $\tau_{T,i}$ is small because δ_z is small in the tunneling region.

Moreover, the imaginary part of the time is then the characteristic time of evolution of the wave function, in the sense brought by Messiah [60] (see discussion in [14]), and also could be present after the tunneling and not only inside the barrier.

In the PPT model, used by Torlina et al and others, the time $t_s = t_R + i t_T$ or the solution of the saddle point equation, defines the starting point of the (semi-classical) trajectory [65]. An imaginary time component to describe the T-time seems to be a result of the external time frame. A dynamical time should be connected to the system, i.e. one should use an internal clock. The solution of the saddle point equation is complex (i.e. of the form $Re(t) + Im(t)$) but this did not means, that the T-time is the imaginary part of the solution and the real part is the ionization time after tunneling. In fact, Perelomov et al stated in their work [38] that the ionization time at atomic field strength F_a occurs in a time comparable with the atomic time, which is real and should serve as an initial point or the reference point (τ_{F_a} or $\tau_{T,i}$) of the attoclock, i.e. for the counting of the T-time or the ionization time in attosecond experiment, because at this field strength the barrier width equals zero (no barrier), and the barrier appears for $F < F_a$, as shown in our model, it increases gradually and becomes infinite for $F \rightarrow 0$, where the T-time becomes infinite [10] (tunneling is not possible or the tunneling probability is zero) for the unperturbed system (ground state) as it should be.

Finally, unfortunately there is still no experimental data available for the Hydrogen (or a Hydrogen-like) atom [70], which is similar to the KASE for He, that makes it possible to compare the different time approximations with an experimental finding and clarify (some of) the controversial issue of the T-time, especially the above mentioned points, keeping in mind that the two quantum mechanical concepts (of Kullie and Torlina et al) could be quantitatively equivalent or belong to two physically equivalent gauge pictures, similarly to other quantitatively equivalent treatments, such as the FPI treatment of [9] or the entropic real T-time of [46].

C. The different regimes

Despite the different views concerning the tunneling process, it is still puzzling and rather not well understood. In the SFA one usually assumes a photoelectric effect mechanism or multiphoton absorption (multiphoton processing), it is important in the regime of large Keldysh parameter $\gamma_k > 1$ and also is the usual process in quantum optics experiments, although in quantum optics (where the field strength is weak) usually an operational point of view is used [71, 72], based on the second quantization formalism, for the theoretical treatments and understanding of the experimental findings. It is initiated by Rony Glauber [72] [A photon is what a photodetector detects], which is in line with the operational time-of-arrival concept mentioned above.

Taking into consideration the view of Collins [49] as discussed above, we prefer, see sec III, another point of view in the region of small Keldysh parameter $\gamma_k < 1$ (the tunneling regime), and argue that a scattering mechanism (or elastic collision) is involved in the tunneling process in attosecond experiments such KASE, where a large number of photon are involved, which means that the electron recoil due to the scattering with LWP, with a drift along the radiation direction [73, 74]. It is worthwhile to mention that even the Compton effect can be explained by a semiclassical nonrelativistic approximation, Schrödinger [75], see [76] p. 222-225, where the electron is described by a wave function. Indeed, the experimental investigation of [73, 74] showed a type of nonlinear Compton scattering at high laser intensities, where many photons participate in a single scattering. Theoretically, an earlier work of Eberly [77] proposed an experiment with high photon density $\rho_L (Watt/cm^3)$ in the optical frequency range ω , where the effect to be observed is that the photons of the laser beam interact (collectively) simultaneously with the electron, and give up momentum and energy to the electron depending on a nonlinearity dimensionless parameter $0 < \varepsilon \sim \rho_L/\omega < 1$.

This is important because it also brings up the particulate nature of the radiation in the discussion, although Lamb, in a nice paper titled “anti-photon” [21], concluded and argued to give up the photon as a particle [It is high time to give up the use of word “photon”, and of a bad concept which will shortly be a century old. Radiation does not consist of particles, and the classical, i.e. non-quantum, limit of quantum theory of radiation is described by Maxwell’s equations for the electromagnetic fields, which do not involve particles.] Ironically, Einstein himself did not accept wave-particle duality (WPD). He wrote: this interpretation... (WPD) .. appears to me as

only a temporary way out [78]. Although, in the earlier days of quantum mechanics, Duane [79] was able to explain diffraction of x-rays by a crystal in terms solely of their particle aspect.

However, modern experiments of quantum optics have beautifully confirmed the corpuscular character of the photon, Zeilinger [22, 80]. Furthermore, Zeilinger concluded that the results suggest that one has to abandon at least the notion of naïve realism that particles have certain properties that are independent of any observation (pre-existing properties). We will rely on this conclusion of Zeilinger and see that the KASE, is a case (and generally in the ASEs) to realize the existence of a corpuscular property of the (attosecond pulse) radiation, precisely of an attosecond LWP.

III. THE TUNNELING PROCESS

A hallmark of the tunneling regime in strong field ionization is the Keldysh parameter γ_K , eq 1. For $\gamma_K < 1$ it follows that

$$\gamma_K = \frac{\sqrt{2I_p}}{F} \omega_0 < 1 \Rightarrow \left(\frac{F}{\omega_0}\right) > \sqrt{2I_p} = k_e \quad (5)$$

where k_e is the wave vector or the momentum of the electron. The left hand side of the second inequality of the relation (5) can be interpreted as an average momentum of the laser pulse $p_L = k_L$. This leads to the following:

$$\left(\frac{F}{\omega_0}\right) = k_L = M_L c, \text{ with } M_L = \left(\frac{k_L}{c}\right) = \left(\frac{F}{c\omega_0}\right) \quad (6)$$

where M_L is an "effective" average mass (or simply effective mass), a feature of the whole photonic laser pulse or the LWP, c is the speed of light and ω_0 (in *au*) is the photon energy or the central circular frequency of the LWP. M_L has nothing to do with the rest mass of a single-photon, which has an upper limit of $m_{ph} \leq 10^{-49} g \approx 10^{-17} eV$ [81]. Indeed, different authors suggested a mass characteristic for the light pulse, where experimentally it was found that the light pulses propagate in the vacuum with a speed somewhat smaller than the speed of light c [82]. Fedorov [83] claims that a nonzero invariant mass of

$$m_{Fedorov} = N \frac{\lambda_0}{2\pi w_a} \frac{\hbar\omega_0}{c^2} \quad (7)$$

can be attributed to the light pulse, where $w_a \gg \lambda_0$ is the pulse width (waist) at $z = 0$, at the center where a Gaussian pulse becomes approximately a plane wave, $\omega_0 = ck_0$ the central frequency of the laser pulse and N the photon number. $m_{Fedorov}$ characterizes global features of the pulse as a whole. It is astonishing to see that for $\lambda_0 = 2\pi w_a$ one gets $m_{Fedorov}/N = m_{ph} = E/c^2 = \hbar\omega_0/c^2$. The invariant mass

of pulses is shown to be related directly with the propagation velocity of pulses in vacuum, which is found to be smaller than the light speed in all cases except an infinitely extended plane wave [81, 83].

The effective mass M_L differs from the "sum" of all the single-photon masses $\bar{n}_{ph} \cdot m_{ph} = \bar{n} \frac{\omega_0}{c^2}$, where \bar{n}_{ph} is the mean number of the photon in the pulse, in this case m_{ph} is obtained from the relation $E = m_{ph} c^2$ according to the theory of relativity. $m_{ph} = E/c^2 = \hbar\omega_0/c^2$ has been introduced formally as an effective mass entering a formal Schrödinger equation for a "photon-wave function", the field amplitude F [76] page 153. It is clear that a wave packet is not a single-photon wave and only in the limit of a plane wave, is ω_0 the energy carrier of a single photon (where the photons density related to the intensity $I_L \sim F^2$ via I_L/ω_0 .) We can consider the LWP as a carrier of a finite number of energy quanta, the (average) number of photons in the pulse. Similarly, on the particle side a plan matter-wave corresponds to a particle stream and a single particle is described by a matter-wave packet (MWP). At this point I have to stress not to anticipate any possible misinterpretation of my claim in this section. There is no reason to interpret M_L (or m_{ph}) as a mass of a small bullet (point-particle) or a group of point-particles. However, light quanta or the photon can be considered as a particle in the sense that it shows particle behavior as claimed by Grangier et al [84]. And according to Compton [85] (see [76], page 224): one concludes light can also consist of discrete units moving in certain directions, each unit has the energy $\hbar\omega$ and the impulse $\hbar\omega/c$. Hence for a radiation of LWP with an approximate intensity $I_L \approx \omega_0 = \omega_{ph}$, $n_{ph} \approx 1$, one obtains

$$\lim_{I_L \approx \omega_0} M_L \approx m_{ph} = \omega_{ph}/c^2 \Rightarrow F_{ph} = \omega_{ph}^2/c = E_{ph}^2/c \quad (8)$$

Although it is difficult to interpret this relation, but a possible simple interpretation would be, F_{ph} is a limit of the electric field strength (for a fixed circular frequency ω_0) for which a LWP can approximately be considered as a single-photon pulse, in the sense that it can show particle behavior. A similar situation when the De Broglie wave length becomes smaller than the geometry of the particle. For the experimental setup of the attosecond experiment, in the optical range, this is a very small value ($F \sim 10^{-5} au$), the typical field strengths used in the ASEs $\sim 0.01 - 0.11 au$. In the context of the field quantization, according to Purcell [86] (see [76], page 153), it makes no difference whether we think of $\rho = \hat{F}^2$ as a square of the electric field strength or the photon probability density ρ .

Now we can turn back to eq 5 keeping in mind the above discussion, especially that our concern is the impact of the

LWP as a carrier of momentum, when it interacts with the electron, where LWP is a group of light quanta or photons propagating in the vacuum with a speed v_c slightly smaller than the speed of light c [82, 83], and showing a particle-behavior. Any further interpretation is beyond the scope of this work. Eq 5 has a simple interpretation in the language of QM, for $k_L < k_e$, ($\gamma_K > 1$) the ionization happens through multiphoton absorption (MPI regime). Whereas for $k_L > k_e$, ($\gamma_K < 1$) the tunneling regime, one figures out that, as the average LWP momentum is larger than the electron momentum, the tunnel ionization happens by a heavy scattering of the (bound) electron through the LWP. The LWP acts collectively and strongly and the electron is therefore, not able to form immediate metastable or virtual states, as it is the case in the MPI regime. The electron response happens on a fast time scale. Consequently, and unlike the MPI regime, the ionization depends mainly on the field strength, and not significantly on the frequency (which is significant for the MPI) [87] chap. 1, and the electron rather follows the value of the laser intensity (the intensity envelope of LWP) [87] chap. 2.3, see below, thus the electron undergoes a dynamical transport [49], see sec II B. Conceivably, which is justified by Collins [49], the scattering time concept (the decay of metastable states, compare subsec II B), when used to calculate the T-time in the ASEs, has no success in cases such as KASE as shown by Landsman [9, 50] and already mentioned in subsec II B.

In accordance with the view of Collins, subsec II B, we suggest a scattering mechanism (or elastic collision) in the tunneling regime as mentioned above subsec II C, it is not that of a single photon scattering (Compton scattering), but that the LWP collectively scatters the electron. This collective process shows a particulate nature of the LWP, similar to the way the collective process of the particles shows the wave behavior on the screen in the double slit experiment (DSE). The latter happens even when one sends the electron one-by-one [88],[89] (voted the most beautiful experiment by readers of Physics World in 2002, Sep 1.) For the former similar experiments can be performed, like the single-photon double-slit experiment [84], or the Compton scattering experiment [90, 91] with single-photon LWP provided is experimentally achievable. Zeilinger [22] concluded that the results of these (achieved) experiments confirm that the quantum state is not just a statistical property of an ensemble of particles. The conceptual questions arising for photon interference are the same as those arising for interference of massive particles, and in both cases we see particle-like and wave-like properties. And that inequivalence for certain interference experiments arises because the photon

has no rest mass [22]. Clearly, the De Broglie wave relation $\lambda = 1/p$, with p the momentum, is the basis for the wave nature of a particle and the basis for the photon hypothesis as a carrier of a momentum $k = m_{ph}c = \omega/c$ [92] (De Broglie Nobel-prize lecture), see eq 10 below, where m_{ph} here is usually interpolated as the “motional” mass of the photon [90]. However, one has to avoid a misinterpretation of m_{ph} because the rest mass of the photon is zero ($< 10^{-49}g$) [81].

In table I the effective mass M_L for a range of intensities used in the ASEs is given. The average number of photons $\overline{n_{ph}}$ is calculated with the assumption that the laser pulse hits the electron in its ground state orbital (1s for He atom, with the radius $r_e \approx 1au$), with the cross-section $A_e = 4\pi r_e^2 = 4\pi$, the area of a sphere of the ground state during the time period of one cycle $\delta t_e = 1/\sqrt{2}I_p$ (i.e. such that the probability amplitude of finding the electron on a spherical shell of the ground state during the time δt is $\psi^2 = 1$.) The average number of photons is then $\overline{n_{ph}} = (I_L/\omega_0)A_e\delta t_e$. Assuming a photon Compton scattering, see [90] and [91], the momentum transmitted to the electron is $\sim (\frac{\omega_0}{c})$ [85], which is too small (as expected) relative to the momentum of the electron k_e , $\frac{\overline{p_C}}{k_e}$ is given in percent in table I, where $\overline{p_C} = \overline{n_{ph}}(\frac{\omega_0}{c})$. Note that a semiclassical approximation due to Schrödinger [75] explains the Compton effect. Whereas assuming a collective LWP scattering, the average momentum transmitted to the electron is $\overline{p_W}$. The quantity $\frac{\overline{p_W}}{k_e}$, given in table I in percent relative to the momentum of the electron, is calculated as the following, see A:

$$\overline{p_W} = \eta \alpha \left(\frac{F}{\omega_0} \right)^2, \quad (9)$$

where $\alpha = 1/c$ is the fine structure constant, which is equal to the strength of the interaction of the photon with the electron. Note that the interaction of an electron with an intense laser field is characterized by $(F/\omega_0)^2$ [73] (and ref 10 inside it.)

In eq 9 the quantity $\overline{p_W}$ determines the amount of an average momentum that is transmitted to the electron, and is (depending on the unknown parameter η) much larger than $\overline{p_C}$ of Compton scattering as seen in table I. One notes that in the above consideration $\overline{p_W}/\overline{p_C} \sim \overline{n_{ph}}^{-1}(F^2/\omega_0^3) = \eta(\delta t A_e \omega_0^2(1 + \epsilon^2)^{-1} = \eta \cdot 15.89$ (ω_0, ϵ are given in sec I). It is worthwhile to mention that in strong field experiments, the process of scattering in the tunneling and the MPI regimes are complex and nonlinear [87]. As the process is nonlinear, it is possible that η depends on the mean photon number $\overline{n_{ph}}$ involved in the interaction (not the total number of the photons of the LWP), which depends on the probability density to find a photon in the interaction volume/area, compare $m_{Fedorov}$

x^a	1	2	3	4	5	6	7.5
$I_n(au)$	0.00285	0.0057	0.00855	0.0114	0.01425	0.0171	0.0214
$F(au)$	0.040	0.057	0.07	0.081	0.090	0.099	0.11
M_L	0.0048	0.0067	0.008	0.0095	0.0106	0.0116	0.013
$\overline{n_{ph}}$	0.43	0.86	1.3	1.72	2.15	2.6	3.23
$\frac{\overline{p_C}}{k_e} \%$	0.0145	0.023	0.043	0.058	0.072	0.087	0.11
$\frac{\overline{p_W}}{k_e} \%$	0.23	0.46	0.69	0.92	1.15	1.38	1.72

TABLE I. ^a Intensity = $x * 10^{14}$ in Wcm^{-2} , I_n = intensity, F = field strength, $\overline{n_{ph}}$ = mean photonic number, $M_L = \alpha(\frac{F}{\omega_0})$, $\overline{p_C} = \overline{n_{ph}} \alpha \omega_0$, $\overline{p_W} = \eta \alpha F^2 / \omega_0^2$, $\eta = 1$, see text.

eq 7, which depends on N .

Hence the determination of a precise value of η needs further investigations, perhaps using wave packet dynamics. Indeed, it will also be fruitful to achieve a Compton-type experiment using strong field attosecond isolated short pulses, e.g. LWP in the visible light, UV or XUV range. Today the experimentally achievable pulse durations are sufficiently short, about 67 – 130 as [4, 93]. Note eq 9 is related to U_p/c , where U_p is the well known ponderomotive energy and is defined as the cycle-averaged quiver energy of the electron in the laser field. For a free electron $U_p = \chi F^2 \omega_0^{-2}$, $\chi = 1/2$ (for linear polarized laser field).

In MPI regime the average photon momentum is not large enough to destabilize the electron in its circular movement, and the electron is captured in a metastable state (by absorbing a photon) with a new wave vector $k'_e < k_e$ and a new oscillation frequency (e.g. $\omega'_e = \omega_e - \omega_0 < \omega_e$). In the case of virtual states the electron evolves successively through the virtual states to a final orbital or to the continuum by absorbing portions of energy (many photons) from the radiation. Einstein's insight to the Planck hypotheses was to see that absorbing energy from radiation in a quantized form leads to the fact that light implies a quanta (photon), which carries the energy ω_0 , although the MPI process, and the photoelectric effect itself, is not the regime, where the corpuscular property emerges from the radiation or from a laser pulse, i.e. no need for the quantization of the classical electromagnetic radiation (only quantization of the matter) [37]. Similarly the Compton effect can be explained by a semiclassical approximation as mentioned above [75]. In the attosecond experiment we see the advantage that in the same experiment, both properties of the radiation can emerge, the wave property (in MPI regime, no photon-particle concept needed) and the corpuscular property (having particle-behavior) in the tunneling regime. Roughly the range of γ_K determines the different regimes. Likewise

the DSE with light (Young experiment) is used to prove the wave nature of light and the DSE with matter is used to prove the wave nature of particles. That the KASE (in the tunneling regime) is also a DSE, was discussed in [14] in accordance with the use of the Feynman path integral by Landsman [9] to calculate the T-time. Hence, the wave/particle duality nature of the radiation can be impressively shown in the ASEs such as the KASE, with the caution that the present work presents only a step in this direction, and certainly, the experiment will enlighten and is decisively important to prove our approach. We discuss this further by drawing the following remarks to our attention.

- The wave packet concept represents an unifying mathematical tool that can cope with and embody nature's particle-like behavior and also its wave-like behavior, Zettili [94], chap 1.8.
- M_L is an effective mass of the LWP, which acts collectively (like a MWP) at a time scale much smaller than $1/\omega_0$.
- The corpuscular property can be best judged from the momentum, $p_{photon}, p_{particle} = \hbar k$. For a stream of photons or a plane wave (the "motional" mass of a photon $m_{ph} = \omega_0/c^2$) $\overline{m_{ph}} \sim \overline{n_{ph}} m_{ph} = \overline{n_{ph}} \omega_0/c^2$, where $\overline{n_{ph}}$ is the average number of the photons. For a LWP $M_L = k_L/c = \frac{F}{c \omega_0}$
- From eq 5, $k_L > k_e$ or $c M_L > k_e$, where $M_L = k_L/c$, the corpuscular nature (particle-behavior) of the LWP is a reasonable assumption, because the De Broglie relation is based on the symmetry between light (wave-light) and particles, as is clear in the following relation [92]

$$\begin{cases} m_{ph} c = \frac{E}{c} = \frac{\omega_{ph}}{c} = p & \text{the particle nature} \\ E = c k, \lambda_D = \frac{1}{p} & \text{the wave nature} \end{cases} \quad (10)$$

De Broglie wrote: I was guided by the aim to perform a real physical synthesis, valid for all particles, of the coexistence of the wave and of the corpuscular aspects that Einstein had introduced for photons in his theory of light quanta in 1905 [95]. Naturally the symmetry in eq 10 permits both directions, wave property for particles or MWP and a corpuscular property of the radiation or LWP. Indeed, De Broglie insight was to see that for the light quanta from $E = h\nu = hc/\lambda$ and $E = cp$ it follows that $\lambda = h/p$ and the same holds for the particles, where h is the Planck constant.

Today the question is clear: under which conditions of an experimental setup or an observation, is a property reflected. Clearly this point of view did not contradict the operational point of view, which circumvents the natural (internal) process of detecting the photon. But additionally, the former considers the fact that an instrumental observation, measuring quantities, reflects a certain property as mentioned above [22, 80], see sec II C last paragraph. The famous DSE shows the wave nature of the particles, the modern quantum optics experiments show the particulate nature of the radiation, and similarly in the KASE-type experiment we encounter the corpuscular property of LWP. In a related issue, as mentioned above, Fedorov [83] showed that the invariant mass of pulses (LWP) eq 7, characterize a global feature of the pulse and is related directly with the propagation velocity of pulses in the vacuum. In the LWP, it is easy to figure out that the average momentum is a collective effect (M_L) and is equivalent to the DSE for particles, in which the collective effect of the particles shows the interference (wave-property) picture on the screen. Hence the question is, when using a “single-photon” LWP, would it show the same effect (for $\gamma_K < 1$), and how similar is it to the Compton effect [90]. It is instructive to mention at this point that in 1986 Grangier et al [84] reported a modern laser based version of Taylors experiment [96]. They provide convincing evidence, that with a suitable care one can prepare single-photon states of light. Such photon states, when sent to a beam splitter, display the type of statistical correlations we would expect of particles. In particular the single photon appears to go one way or the other. Yet such single-photon states can interfere with themselves, even when run in “delayed choice” [97, 98], in a way similar to the particles in the DSE [76] p. 213.

To conclude, the situation in the attosecond experiment, such as the KASE, in which the collective act of the LWP shows (for $\gamma_K < 1$) a particulate property k_L, M_L , is similar to the DSE, in which the collective act of the particles shows a wave property, even when the particles are sent to the screen one-by-one [22, 88]. And similarly, in the case of a particulate property for photons [84, 98] (see also [76, 99] p. 153), the interference depends on the number of the photons, even when the field strength is so small that the photons reach the screen only one-by-one.

In the MPI region ($\gamma_K > 1$), the evolution of the electron wave packet is under intensive research. Unfortunately, the measurements are restricted to the relative delays between two photo-electron wave packets ionized from two different orbitals, for details the reader is kindly referred to the tutorial

[27]. One notes that the Keldysh time τ_K is large, for in the region of $\gamma_K > 1$ the field strength of the laser pulse is small and does not disturb the electron heavily, the evolution of the electron’s MWP is relatively slow, while for $\gamma_K < 1$ the electron is enforced to move at a fast time scale $\tau_{T,sym}$ (or $\tau_{T,d}$), see eq 2. Furthermore, in the immediate region $\gamma_K \sim 1$, which is narrow [87] (chap 1, p. 2), there is still no clear picture, where both the MPI and tunneling can take place [32, 100] and investigations show that non-adiabatic effects affect the tunneling, i.e. differently from the adiabatic approximation, where the electron sees approximately a static electric field during the ionization process [47, 101–104].

Conclusion We have discussed in this work different points in the ASEs, related to the issue of the tunneling time, the tunneling process and wave-matter duality. The tunneling time $\tau_{T,d}$ differs from the parametric time, which equals the numerical τ_{num} , which is obtained by the numerical integration of the TDSEQ. We proposed a scattering mechanism of the electron by the laser wave packet in the region of the Keldysh parameter $\gamma_K < 1$, and argued that the corpuscular nature (particle-behavior) of radiation emerges in this region, similar to the double slit experiments in quantum optics, where the wave-matter duality of radiation and particles is proved.

Appendix A

Due to the scattering process with the electron, the LWP with an average momentum k_L undergoes a perturbation. We can expand the momentum k'_L in powers of $(\frac{F}{\omega_0}) = k_L$, where k'_L is the average momentum of the LWP after it scatters the electron (F, ω_0 as in sec. III),

$$k'_L = p'_L(\omega_0, F) = \sum_{i=0} a_i k^{(i)} = a_0 k^{(0)} + a_1 k^{(1)} + \dots \quad (A1)$$

The perturbation terms in A1 obey,

$$k^{(i)} = \alpha^i \left(\frac{F}{\omega_0} \right)^{(i+1)}, \quad i = 0, 1, 2, \dots \quad (A2)$$

where $\alpha = 1/c$ (the fine structure constant), c the speed of light. Without loss of generality, we can take $a_0=1$, and

$$a_{i \geq 1} = \begin{cases} 0 & \text{for } k_L = p_L(\omega_0, F) \\ \neq 0 & \text{for } k'_L = p'_L(\omega_0, F) \end{cases} \quad (A3)$$

i.e. for the initial wave packet we set $k_L = k^{(0)} = (\frac{F}{\omega_0})$. The series can safely be truncated to the first order in α (second order in $(\frac{F}{\omega_0})$). The small change in LWP is of the order $k_L - k'_L \sim \alpha \left(\frac{F}{\omega_0} \right)^2$, and is about 1% of $k^{(0)}$. This finally leads to eq 9.

It is worthwhile to mention that, Meyerhofer [73], the traditional boundaries between Thomson and Compton scattering become less clear. For example, nonlinear Compton scattering of photons and an electron takes place in the presence of an intense laser field, mixing quantum mechanical and classical pictures. Hence eq 9, sec. III, is valid for LWP scattering, where η depends on the mean number of the photons involved in the interaction, not on the total number of the photons of the pulse N .

Acknowledgments I would like to thank Prof. Martin Garcia from the Theoretical Physics of the Institute of Physics at the University of Kassel for his kind support.

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